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Walsh–Fourier Series for Functions of n Variables with Localization

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When it is desired to represent a function of n variables by a series of the Fourier type, it is customary to construct the relevant complete orthogonal set by taking a set of functions each of which is a product of n functions of single variables. With Walsh functions, an alternative is possible, feasible, and practically implementable. The principal advantage of the alternative scheme is that localization holds in the same sense as for a function of a single variable represented by a series of the Fourier type. Another feature of the alternative method is that the Walsh–Fourier series for certain types of discontinuous functions behave essentially as if those functions were continuous.

I. INTRODUCTION

Since the publication of Fine's classic papers on Walsh functions [1–3], little interest has been shown in "multiple Walsh–Fourier series" or Walsh function representation of functions of n variables. The Walsh functions, in Paley's definition, designated by

$$\psi(k, r), \qquad (k = 0, 1, 2,...),$$

are defined for $r \in \mathfrak{E}_1$, the real line; since they are of period 1 it is customary to study them on the unit interval

$$U_1 = \{r \mid 0 \leqslant r \leqslant 1\}.$$

$\mathfrak{E}_n$ is n-dimensional Euclidean space. The unit n-cube is

$$U_n = \{\mathbf{x} \mid 0 \leqslant x_i \leqslant 1, i = 1, 2,..., n\}.$$

Let $g(\mathbf{x})$ be a real valued integrable function defined on U_n. In analogy with trigonometric Fourier series, it has been assumed that the appropriate procedure, if a Walsh–Fourier series for g is desired, is to consider series of the form

$$g(\mathbf{x}) \approx \sum a(k_1, k_2, \ldots, k_n)\, \psi(k_1, x_1)\, \psi(k_2, x_2) \cdots \psi(k_n, x_n).$$

The failure to study this system of multiple Walsh functions is partially due to the extreme simplicity of many of the standard problems, given the technique that has been developed for trigonometric Fourier series, both ordinary and multiple, and for ordinary Walsh–Fourier series. However, some problems are relatively difficult, owing to some important differences between Walsh and trigonometric functions. As Fine pointed out, one of the most important differences between ordinary Walsh sreies and ordinary trigonometric series is that the Fejér kernel for the former is relatively ill behaved.

The most striking difference between ordinary and multiple Fourier series, whether of Walsh or trigonometric functions, is that the principle of "localization" does not hold in the latter case. This fact is a direct consequence of the product form

$$\psi(k_1, x_1)\, \psi(k_2, x_2) \cdots \psi(k_n, x_n) \tag{1}$$

(allowing the ψ, temporarily, to be either trigonometric or Walsh functions), and is inescapable without making exceedingly strong assumptions about g, such as that it is bounded on U_n [9, Chap. 17], which is just another way of saying that the behavior of a multiple Fourier series in a neighborhood may depend on the nature of g at points remote from the neighborhood.

This paper presents an alternative Walsh system for representation of functions of n variables. Since the new system is not a "multiple Walsh–Fourier series," i.e., the product form (1) is not employed, localization holds and the localization theorems are not only essentially similar to those of the single variable case, but follow directly from them. For the sake of clarity, the presentation is relatively informal.

II. Walsh Functions

Let $f(r)$ be Lebesgue integrable on [0, 1] and periodic with period 1 (thus, if f were further specified as continuous, it would be necessary that $f(0^+) = f(1^-)$). The Walsh–Fourier coefficients are

$$a_k = \int_0^1 f(r)\, \psi(k, r)\, dr,$$

the partial sum

$$s_m(r) = \sum_{k=0}^{m-1} a_k \psi(k, r),$$

and the Cesàro sum

$$\sigma_M(r) = \frac{1}{M} \sum_{m=1}^{M} s_m(r).$$

The following are important properties of the Walsh functions in respect to Walsh–Fourier series:

1. $a_k \to 0$ as $k \to \infty$.

2. Define the moduli of continuity

$$\mu_0(\delta, f) = \sup_{|h| \leqslant \delta} |f(r + h) - f(r)|,$$

$$\mu_1(\delta, f) = \sup_{|h| \leqslant \delta} \int_0^1 |f(r + h) - f(r)|\, dr.$$

Then, for $k > 0$

$$| a_k | \leqslant \mu_0(1/k, f)/2,$$
$$| a_k | \leqslant \mu_1(1/k, f).$$

As a corollary, if $f \in \mathrm{Lip}\, \alpha$, $0 < \alpha \leqslant 1$, then

$$a_k = \mathcal{O}(k^{-\alpha}).$$

3. If f is of bounded variation on (0, 1) and D is its total variation over (0, 1), then

$$| a_k | \leqslant \frac{D}{k}, \qquad k > 0.$$

4. If f is absolutely continuous and $a_k = o(1/k)$, then $f =$ constant.

5. If f is continuous at r, then $\sigma_M(r) \to f(r)$. If f is continuous, $\sigma_M \to f$ uniformly. If $\alpha > 0$, $s_m(r)$ is (C, α) summable to f a.e.

6. (i) $s_{2^m}(r) \to f(r)$ a.e. as $m \to \infty$, in particular, at every point of continuity.

(ii) If f is continuous in an interval (r_1, r_2), then $s_{2^m}(r) \to f(r)$ uniformly as $m \to \infty$, in any subinterval $[\rho_1, \rho_2]$, where $r_1 < \rho_1 < \rho_2 < r_2$.

(iii) If f is of bounded variation and r is either a point of continuity or a dyadic rational, $s_m(r)$ converges.

(iv) If f is of bounded variation and r is neither a point of continuity, nor a dyadic rational, then $s_m(r)$ diverges.

(v) For any r, there exists a continuous function whose series diverges at r.

(vi) If f is continuous and

$$\mu_0(\delta, f) = o(\log \delta^{-1})^{-1}$$

as $\delta \to 0$, then $s_m \to f$ uniformly. Thus, if $f \in \text{Lip}\ \alpha$, $\alpha > 0$, $s_m \to f$ uniformly.

(vii) If f is continuous and of bounded variation on [0, 1], $s_m \to f$ uniformly (both (vi) and (vii) can be broadened somewhat).

(ix) There exists an L^1 function such that s_m is divergent a.e.

(x) If f is an L^2 function, $s_m \to f$ a.e.

7. (i) If s_m converges to zero uniformly except in the neighborhood of a finite number of points (i.e., if there exists a set of points $r_1, r_2, \ldots, r_N$ such that, for any $\delta > 0$, $s_m \to 0$ uniformly on the set $R = \{r \mid r \in U_1, \mid r - r_i \mid \geqslant \delta \text{ for } i = 1, 2, \ldots, N\}$), then $a_k = 0$ for all k.

(ii) If $s_m \to 0$ except on a denumerable set, then $a_k = 0$ for all k.

(iii) If s_m converges to an integrable function except on a denumerable set, the series is the Walsh–Fourier series of the function.

8. If $f(r) = 0$ in an interval (r_1, r_2), then $s_m(r) \to 0$ uniformly as $m \to \infty$, in any subinterval $[\rho_1, \rho_2]$, where $r_1 < \rho_1 < \rho_2 < r_2$.

The last property is the localization principle for Walsh–Fourier series, essentially as it was expressed in Walsh's original paper. The subject can be considerably elaborated and expanded, as was done by Fine. However, here we treat the subject of localization in the original and simplest sense of the term.

III. Hilbert's Space Filling Curve

Hilbert's space filling curve $\mathbf{h}_n$ [10, Vol. 1, Chap. 5] is a function mapping the unit interval onto the unit n-cube continuously; $U_n = \mathbf{h}_n(U_1)$. It has properties particularly compatible with dyadic number representations, and thus, with Walsh functions.

Regard $n > 1$ as fixed. For an integer $k \geqslant 0$ the set $\mathscr{V}_k$ is the set of 2^{nk} intervals of the form

$$V_{k,m} = \{r \mid (m - 1)\, 2^{-nk} \leqslant r \leqslant m2^{-nk}\},$$

where $m = 1, 2, \ldots, 2^{nk}$. Thus, $V_{k,m} \in \mathscr{V}_k$. Similarly, define the set $\mathscr{W}_k$ of 2^{nk} n-cubes of the form

$$W_{k,p} = \{\mathbf{x} \mid (l_{k,p,i} - 1)\, 2^{-k} \leqslant x_i \leqslant l_{k,p,i} 2^{-k},\ i = 1, 2, \ldots, n\},$$

where p and $l_{k,p,i}$ are integers, $1 \leqslant p \leqslant 2^{nk}$, $1 \leqslant l_{k,p,i} \leqslant 2^k$, and the $l_{k,p,i}$ are such that the collection of 2^{nk} n-cubes $W_{k,p} \in \mathscr{W}_k$ covers U_n.

The principal relevant feature of Hilbert's space filling curve $\mathbf{h}_n$ is that any interval $V_{k,m}$ maps continuously onto some n-cube $W_{k,p}$. For this reason, the $l_{k,p,i}$ can be chosen such that

$$W_{k,m} = \mathbf{h}_n(V_{k,m}). \tag{2}$$

The relationship (2) is obvious from the geometric manner in which Hilbert exhibited his curve (Fig. 1). It is also clear that $\mathbf{h}_n$ is measure preserving.

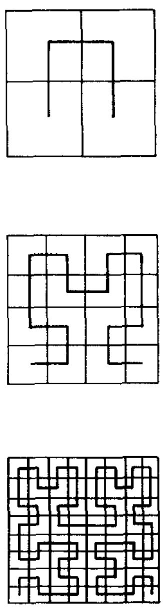

FIG. 1. Hilbert's space filling curve is the limit of this sequence.

A function $g(\mathbf{x})$, $\mathbf{x} \in U_n$, is Lebesgue integrable in U_n iff $f(r) = g(\mathbf{h}_n(r))$ is Lebesgue integrable in U_1; f is termed here the *image* of g.

When $n = 2$, Hilbert's space filling curve is unique except for obvious rotations in U_2. When $n > 2$, such uniqueness does not hold for space filling curves possessing the one to one relationship between intervals $V_{k,m}$ and n-cubes $W_{k,m}$ (the distinguishing feature of Hilbert's curve). One specific

$\mathbf{h}_n$ for general n has recently been studied in detail, and specific algorithms for calculating $\mathbf{x} = \mathbf{h}_n(r)$, given r, were presented [11–13].

Each $\mathbf{x} \in U_n$ has an inverse image in U_1, under the mapping $\mathbf{h}_n$, which consists of at most 2^n points. If no component of $\mathbf{x}$ is dyadic rational, the inverse image is a single point. If $X \subset U_n$, and $R \subset U_1$ is the set of inverse images of points $\mathbf{x} \in X$, then R is called the inverse image of X.

Obviously, the set of points in U_n, each having an inverse image consisting of more than one point, is of measure zero. Thus, given any function $f'(r)$ defined on $\mathfrak{E}_1$, there exists a function $g(\mathbf{x})$, $x \in U_n$, whose image $f = f'$ a.e. in U_1.

Let $\mathbf{0}$ be the n-vector $(0, 0, 0,..., 0, 0, 0)$ and let $\mathbf{u}$ be the n-vector $(1, 0, 0,..., 0, 0, 0)$. The $\mathbf{h}_n$ studied in [11–13] have the property that $r = 0$ is the inverse image of $\mathbf{x} = \mathbf{0}$, and that $r = 1$ is the inverse image of $\mathbf{x} = \mathbf{u}$. Henceforth, we assume that $\mathbf{h}_n$ has this property.

Any function θ_n: $U_n \to U_1$ that has the property

$$\mathbf{x} = \mathbf{h}_n(\theta_n(\mathbf{x})),$$

for all $\mathbf{x} \in U_n$, is termed an *inverse* of $\mathbf{h}_n$.

For any $n \geqslant 2$ and q, $1 \leqslant q \leqslant \infty$, there exist constants $0 < \mathscr{M}_0 < \mathscr{M}_1 < \infty$ such that for any $r \in U_1$ and $\mathbf{x} = \mathbf{h}_n(r) \in U_n$.

(i) For every $r' \in U_1$

$$\| \mathbf{x}' - \mathbf{x} \|_q \leqslant \mathscr{M}_1 \mid r' - r \mid^{1/n},$$

where $\mathbf{x}' = \mathbf{h}_n(r')$.

(ii) For any $\delta > 0$, there exists $r'' \in U_1$ such that $\mid r'' - r \mid < \delta$ and

$$\| \mathbf{x}'' - \mathbf{x} \|_q \geqslant \mathscr{M}_0 \mid r'' - r \mid^{1/n},$$

where $\mathbf{x}'' = \mathbf{h}_n(r'')$.

The one to one relationship between intervals $V_{k,m}$ and n-cubes $W_{k,m}$ immediately implies what we call the finite covering property.

Finite Covering Property

For $\mathbf{x} \in U_n$, let X_2 be an open (in U_n) sphere and let X_1 be a closed sphere centered at $\mathbf{x}$

$$X_1 = \{\mathbf{y} \mid \mathbf{y} \in U_n, \| \mathbf{y} - \mathbf{x} \|_q \leqslant \rho_1\},$$
$$X_2 = \{\mathbf{y} \mid \mathbf{y} \in U_n, \| \mathbf{y} - \mathbf{x} \|_q < \rho_2\},$$

with $0 < \rho_1 < \rho_2 < \infty$. Let R_1 and R_2 be the inverse images of X_1 and X_2, respectively. Then, $R_1 \subset A \subset B \subset R_2$, where A and B are each the union of a finite number of disjoint closed intervals:

$$A = \bigcup_{i=1}^{N} A_i , \qquad A_i = \{r \mid a_{i,1} \leqslant r \leqslant a_{i,2}\},$$

$$B = \bigcup_{i=1}^{N} B_i , \qquad B_i = \{r \mid b_{i,1} \leqslant r \leqslant b_{i,2}\},$$

where

$$b_{i,2} < b_{i+1,1} , \qquad i = 1, 2,..., N - 1,$$

if $N > 1$ and, if $\mathbf{0} \notin X_1$ and $\mathbf{u} \notin X_1$,

$$\begin{aligned} b_{1,1} > 0, \qquad b_{N,2} < 1, \\ b_{i,1} < a_{i,1} < a_{i,2} < b_{i,2} , \qquad i = 1, 2,..., N. \end{aligned} \tag{3}$$

If $\mathbf{0} \in X_1$, conditions (3) must be modified so that $b_{1,1} = a_{1,1} = 0$. Likewise, if $\mathbf{u} \in X_1$, $b_{N,2} = a_{N,2} = 1$.

If $g(\mathbf{x})$ is continuous on U_n, then its image $f(r)$ is continuous on U_1, but f is not necessarily equal to a function that is continuous and periodic on $\mathfrak{C}_1$ (with period 1). For this, it is required in addition that $g(\mathbf{0}) = g(\mathbf{u})$.

If $g(\mathbf{x})$ is equivalent (equal a.e. on U_n) to a functino $g'(\mathbf{x})$ that is continuous on U_n and has the property $g'(\mathbf{0}) = g'(\mathbf{u})$, then its image $f(r)$ is equivalently continuous and periodic (equal a.e. in U_1 to a function that is continuous and periodic on $\mathfrak{C}_1$). The converse, however, is not true, as can be easily shown by example. The particular $\mathbf{h}_2$ of Fig. 1 has the property that if $\mathbf{x} = \mathbf{h}_2(r)$, $r \in U_1$, then $0 \leqslant r \leqslant \frac{1}{2}$ implies $0 \leqslant x_1 \leqslant \frac{1}{2}$, $\frac{1}{2} \leqslant r \leqslant 1$ implies $\frac{1}{2} \leqslant x_1 \leqslant 1$, and $\mathbf{h}_2(\frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$. Thus, a function ($n = 2$)

$$\begin{aligned} g(\mathbf{x}) &= x_2 , \qquad & 0 \leqslant x_1 \leqslant \tfrac{1}{2} , \\ &= 1 - x_1 \qquad & \tfrac{1}{2} < x_1 \leqslant 1, \end{aligned} \tag{4}$$

which has the property $g(\mathbf{0}) = g(\mathbf{u})$, but is not equivalently continuous, has an image f that is equivalently continuous and periodic, for the only discontinuities of the image are due to values taken by $f(r)$ at points in the inverse image of the set $\{\mathbf{x} \mid x_1 = \frac{1}{2}, 0 \leqslant x_2 \leqslant 1\}$.

It is not possible, however, to characterize functions g whose images are equivalently continuous and periodic without making very specific reference to properties of $\mathbf{h}_n$. This is seen by considering the function ($n = 2$) $g'(\mathbf{x}) = x_1$ for $0 \leqslant x_2 \leqslant \frac{1}{2}$, $= 1 - x_2$ for $\frac{1}{2} < x_2 \leqslant 1$, whose image is not equivalently continuous and periodic, although one would normally consider it the same sort of function as the $g(\mathbf{x})$ of (4).

These considerations motivate the following definition. Let $X \subset U_n$, and let $R \subset U_1$ be the inverse image of X. We say $g(\mathbf{x})$ is *image continuous* on X iff its image $f(r)$ is equivalent to a periodic function that is continuous on R (for any $r \in R$, for any $\epsilon > 0$, there exists $\delta > 0$ such that $r' \in \mathfrak{C}_1$ and $|r' - r| < \delta$ imply $|f(r') - f(r)| < \epsilon$; note the specification of $\mathfrak{C}_1$ rather than R). When it is said that $g(\mathbf{x})$ is image continuous, without specification of X, it is implied that $X = U_n$. If it is said that g is image continuous at $\mathbf{x}$, this means that X consists of the single point $\mathbf{x}$.

Continuity on a set X does not imply that $g(\mathbf{x})$ is image continuous on X. For example, let $g(\mathbf{x}) = 0$ in an ϵ-neighborhood of (and including) $\mathbf{x} = \mathbf{0}$ and $g(\mathbf{x}) = 1$ in an ϵ-neighborhood of (and including) $\mathbf{x} = \mathbf{u}$. However, g is not image continuous on either ϵ-neighborhood, since the image of g is not equivalent to a periodic function that is continuous at $r = 0$ or $r = 1$, both of which points are in the inverse images involved. However, we can say:

Let g be continuous. Then, g is image continuous iff $g(\mathbf{0}) = g(\mathbf{u})$.

Let $X \subset U_n$ be open in U_n, and let g be continuous on X. If *either* $\mathbf{0} \notin X$ and $\mathbf{u} \notin X$, *or* $\mathbf{0} \in X$ and $\mathbf{u} \in X$ and $g(\mathbf{0}) = g(\mathbf{u})$, then g is image continuous on X.

IV. Walsh Functions for U_n

Let $\mathbf{h}_n\colon U_1 \to U_n$ be Hilbert's space filling curve and let $\theta_n\colon U_n \to U_1$ be an inverse of $\mathbf{h}_n$. The Walsh functions for U_n are designated ψ_n and are defined for $n \geqslant 2$ thus

$$\psi_n(k, \mathbf{x}) = \psi(k, \theta_n(\mathbf{x})), \qquad k = 0, 1, 2, \ldots .$$

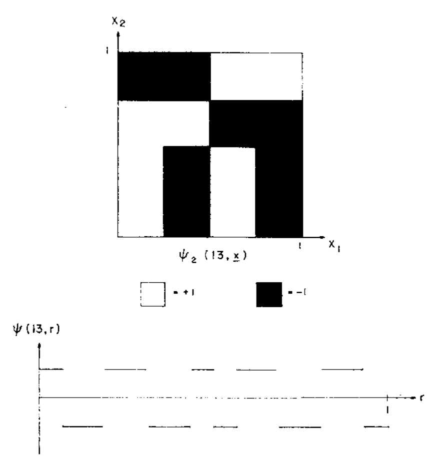

FIG. 2. $\psi_2(13, x)$ and $\psi(13, r)$.

An example is given in Fig. 2, where the relationship between $\psi_2(13, \mathbf{x})$ and $\psi(13, r)$ is illustrated. It is obvious that ψ_2 is not a product of a function of x_1 and a function of x_2. It is also clear that the ψ_n constitute a complete orthonormal set, since if $g(\mathbf{x})$ were orthogonal to all $\psi_n(k, \mathbf{x})$, then $f(r)$, the image of g, would be orthogonal to the ordinary Walsh functions $\psi(k, r)$.

Let $g(\mathbf{x})$ be an integrable function on U_n, and let $f(r)$ be its image. The Walsh–Fourier series for g

$$g(\mathbf{x}) \approx \sum_{k=0}^{\infty} a_k \psi_n(k, \mathbf{x}) = S^{(n)}(\mathbf{x}), \qquad a_k = \int_{U_n} \psi_n(k, \mathbf{x})\, g(\mathbf{x})\, d\mathbf{x}.$$

$S^{(n)}(\mathbf{x})$ may be studied merely by studying the Walsh–Fourier series for f since it is the case that

$$f(r) \approx \sum_{k=0}^{\infty} a_k \psi(k, r) = S(r), \qquad a_k = \int_0^1 \psi(k, r) f(r)\, dr,$$

$$S^{(n)}(\mathbf{x}) = S(\theta_n(\mathbf{x})).$$

The partial sum

$$s_m^{(n)}(\mathbf{x}) = \sum_{k=0}^{m-1} a_k \psi_n(k, \mathbf{x}) = s_m(\theta_n(\mathbf{x}))$$

and the Cesàro sum

$$\sigma_M^{(n)}(\mathbf{x}) = \frac{1}{M} \sum_{m=1}^{M} s_m^{(n)}(\mathbf{x}) = \sigma_M(\theta_n(\mathbf{x})).$$

$G(\mathbf{x})$ is defined as the set of limits of the function g for sequences converging to $\mathbf{x}$, i.e., the real number $\gamma \in G(\mathbf{x})$ iff there exists a sequence $\{\mathbf{x}_i\}$, $\mathbf{x}_i \in U_n$, $\mathbf{x}_i \to \mathbf{x}$, such that $g(\mathbf{x}_i) \to \gamma$ (the values $\gamma = \pm\infty$ are allowed). The expression $\gamma_i \to G(\mathbf{x})$ is interpreted as meaning $\gamma_i \to \gamma$, where $\gamma \in G(\mathbf{x})$. If it is said that a sequence of functions $\gamma_i(\mathbf{x}) \to G$ uniformly on X, this means that there exists a function $\gamma(\mathbf{x})$ such that $\gamma(\mathbf{x}) \in G(\mathbf{x})$ for every $\mathbf{x} \in X \subset U_n$, and that $\gamma_i(\mathbf{x}) \to \gamma(\mathbf{x})$ uniformly on X.

Combining the properties of ordinary Walsh–Fourier series (Section II) with those of $\mathbf{h}_n$ (Section III) one easily arrives at the following properties of $S^{(n)}$:

1′. $a_k \to 0$ as $k \to \infty$.

2′. If $g \in \text{Lip}\,\alpha$, $0 < \alpha \leqslant n$, and $g(\mathbf{0}) = g(\mathbf{u})$, then

$$a_k = \mathcal{O}(k^{-\alpha/n}).$$

5′. If g is image continuous at $\mathbf{x}$, then $\sigma_M^{(n)}(\mathbf{x}) \to G(\mathbf{x})$. If g is image continuous, then $\sigma_M^{(n)} \to G$ uniformly on U_n. If $\alpha > 0$, $s_m^{(n)}$ is (C, α) summable to G, a.e.

6′. (i) $s_{2^m}^{(n)} \to G$ a.e. as $m \to \infty$, in partciular, at every point $\mathbf{x}$ where g is image continuous.

(ii) Let $X_{2,i}$ designate an open sphere in U_n and $X_{1,i} \subset X_{2,i}$ a closed subsphere of $X_{2,i}$. Let sets X_2 and X_1 consist of finite unions of such spheres:

$$X_2 = \bigcup_{i=1}^{N} X_{2,i}, \qquad X_1 = \bigcup_{i=1}^{N} X_{1,i}.$$

Assume that either $\mathbf{0} \notin X_1$ and $\mathbf{u} \notin X_1$, or $\mathbf{0} \in X_1$ and $\mathbf{u} \in X_1$. If g is image continuous on X_2, then $s_{2^m}^{(n)} \to G$ uniformly on X_1. This property is a direct consequency of property 6(ii) of Section II and of the finite covering property.

(v) For any $\mathbf{x} \in U_n$, there exists an image continuous function whose series diverges at $\mathbf{x}$.

(vi) If $g \in \text{Lip}\,\alpha$, $\alpha > 0$, and $g(\mathbf{0}) = g(\mathbf{u})$, then $s_m^{(n)} \to g$ uniformly.

(viii) If $g \in \text{Lip}\,\alpha$, $\alpha > n/2$, and $g(\mathbf{0}) = g(\mathbf{u})$, then $s_m^{(n)}$ converges absolutely.

(ix) There exists $g \in L^1$ such that $s_m^{(n)}$ is divergent a.e.

(x) If $g \in L^2$, $s_m^{(n)} \to g$ a.e.

7′. (i) If $s_m^{(n)}$ converges to zero uniformly in U_n except in the neighborhood of a finite number of points, then $a_k = 0$ for all k. This is a direct consequence of property 7 of Section II, and of the continuity of $\mathbf{h}_n$.

(ii) If $s_m^{(n)} \to 0$ except on a denumerable set, then $a_k = 0$ for all k.

(iii) If $s_m^{(n)}$ converges to an integrable function except on a denumerable set, the series is the Walsh–Fourier series of the function.

8′. Let $X_{2,i}$ designate an open sphere in U_n and $X_{1,i} \subset X_{2,i}$ a closed subsphere of $X_{2,i}$. Let sets X_2 and X_1 consist of finite unions of such spheres:

$$X_2 = \bigcup_{i=1}^{N} X_{2,i}, \qquad X_1 = \bigcup_{i=1}^{N} X_{1,i}.$$

Assume that either $\mathbf{0} \notin X_1$ and $\mathbf{u} \notin X_1$, or $\mathbf{0} \in X_1$ and $\mathbf{u} \in X_1$. If $g = 0$ on X_2, then $s_m^{(n)} \to 0$ uniformly on X_1.

The last property is a direct consequence of property 8 of Section II and of the finite covering property. It is the localization property for Walsh–Fourier series of the new functions, and appears to be the major justification for proposing these functions. Note that localization holds in essentially the same sense as it does for ordinary Walsh–Fourier series of a single variable. That the behaviors of $S^{(n)}(\mathbf{x})$ at the points $\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{u}$ are mutually dependent does not compromise this claim, since the same sort of situation holds for $S(r)$ for the points $r = 0$ and $r = 1$.

V. Remarks

A certain amount of selectivity was practiced in organizing the list of properties noted in Sections II and IV. Although the properties chosen are obviously not near to being exhaustive in respect to what is known about Walsh functions, we feel that they were an appropriate selection for this paper.

Some of the properties of ordinary Walsh–Fourier series that are listed in Section II are not reflected in analogs among the properties of $S^{(n)}$ that are listed in Section IV. Specifically, the concept of the modulus of continuity of a function $f(r)$, the concept of a function $f(r)$ of bounded variation, and the concept of an absolutely continuous function $f(r)$, failed to be reflected in analogous properties of $g(\mathbf{x})$. This failure is certainly not due to the impossibility of finding analogs of the relevant theorems listed in Section II. Indeed, the generation of the analogs would be almost trivial. They have not been listed here, partially on account of their near triviality, and partially because they entail such peculiar manners of regarding functions defined on U_n that it was felt that the appropriateness of making the necessary definitions, and stating the relevant properties, was somewhat questionable. For the same reasons, we have avoided stating analogs of the more elaborate features of Fine's theory of localization.

It is conceded that a similar objection could be made to the concept of "image continuity," likewise a peculiar manner of regarding a function $g(\mathbf{x})$. However, there are two reasons why it is believed that the introduction of the concept of image continuity is appropriate. First, except for the qualification involving the points $\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{u}$, ordinary continuity implies image continuity. By contrast, no ordinary description of a function $g(\mathbf{x})$ leads to the modulus of continuity of its image, the bounded variation of its image, or the absolute continuity of its image. Second, image continuity has practical significance. This is illustrated by the $g(\mathbf{x})$ of (4), a discontinuous function that, however, since it is image continuous, has a Walsh–Fourier series that behaves essentially as if g were continuous. One may then proceed to observe, e.g., that the image f is equivalent to a function in Lip $\frac{1}{2}$, and thus, $a_k = \mathcal{O}(k^{-1/2})$. The general nature of the practical possibilities involved here is obvious. No such possibilities arise with multiple Walsh–Fourier series.

There appear to be no problems of practical implementation, associated with the series $S^{(n)}$, that are not also present with ordinary multiple Walsh–

Fourier series. Practical use of the new functions would often require calculation of $\mathbf{h}_n(r)$ or $\theta_n(\mathbf{x})$, or both, but this is not believed to be a significant drawback. Fast Walsh–Fourier transform algorithms, of the Cooley–Tukey type, may be applied to the image function $f(r)$ without modification.

It is hardly necessary to point out that similar constructions are possible with trigonometric functions. Functions of the form $\exp(\imath 2\pi k \theta_n(\mathbf{x}))$, where $\imath = (-1)^{1/2}$, constitute a complete orthogonal set for U_n and the corresponding Fourier series would have localization properties. However, functions of this sort are relatively badly behaved when compared to ordinary trigonometric functions; they are not likely to be satisfactory for representation of functions $g(\mathbf{x})$ in cases where ordinary trigonometric fnuctions are considered appropriate, especially on account of their abundance of jump discontinuities. The "Walsh functions" $\psi_n(k, \mathbf{x}) = \psi(k, \theta_n(\mathbf{x}))$, by contrast, do not have any general qualitative feature not possessed by ordinary Walsh functions, and thus, would most likely be satisfactory in cases where ordinary sorts of Walsh representations are considered satisfactory.

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